Effects of Transverse Shearing Flexibility on Postbuckling of Plates in Shear

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This paper presents buckling and postbuckling results for plates loaded by in-plane shear. The buckling results have been plotted to show the effects of thickness on the stress coefficient for aluminum plates. Results are given for various length-to-width ratios. Postbuckling results for thin plates with transverse shearing flexibility are compared to results from classical theory. The problems considered are the postbuckling response of plates in shear made of aluminum and of a ±45 deg graphite-epoxy laminate. Thus, the materials are isotropic and orthotropic, respectively. The plates are considered to be long with side edges simply supported, with various in-plane edge conditions, and the plates are subject to a constant shearing displacement along the side edges. Characteristic curves presenting the average shear stress resultant as a function of the applied displacement are given. These curves indicate that change in in-plane edge conditions influence plate postbuckling stiffness and that transverse shearing is important in some cases.

Nomenclature

$A_{11},A_{12},A_{22},A_{33}$, A_{44},A_{55},A_{66}	= orthotropic plate extensional stiffnesses
a,b,h	= dimensions of rectangular plate parallel to x, y, and z axes, respectively
$C_{11},C_{12},C_{22},C_{33}$, C_{44},C_{55},C_{66}	= stiffnesses in Hooke's law for each lamina
$D_{11},D_{12},D_{22},D_{66}$	= orthotropic plate bending stiffnesses
E	= Young's modulus for material
M_y	= bending moment in plate per unit length
N_x,N_y,N_{xy}	= in-plane stress resultants in plate
N_{xz},N_{yz}	= transverse stress resultants in plate
N_{xy}^{av}	= average shear load
N_{xy}^{cr}	= value of N_{xy}^{av} at buckling
U	= applied end shortening
$\bar{u}$	= applied shear displacement
u,v,w	= displacements in x, y, and z directions, respectively
x,y,z	= plate coordinates
$\epsilon_x,\epsilon_y,\epsilon_z$, $\gamma_{yz},\gamma_{xz},\gamma_{xy}$	= strains in plate
λ	= buckle half-wavelength
μ	= Poisson's ratio for material
$\sigma_x,\sigma_y,\sigma_z$, $\tau_{yz},\tau_{xz},\tau_{xy}$	= stresses in each lamina

Introduction

FOR linear problems, classical plate theory predicts inplane stresses and deformation that are comparable to those given by three-dimensional elasticity for thin plates of homogeneous material. Conventional transverse shearing theory makes similar predictions for sandwich construction and for thicker plates of homogeneous material. Transverse stresses are generally small compared to the largest in-plane stress; however, they can be important when the plate is relatively weak in the transverse direction and when the plate response is sensitive to the transverse stiffness as in buckling and the higher modes of vibration.

Classical plate theory and conventional transverse shearing plate theory are two-dimensional and they predict in-plane stresses directly, but they are not accurate enough to predict transverse stresses directly. An accurate nonlinear two-dimensional theory for laminated and thick plates with three-dimensional flexibility has been derived in Ref. 1. The essential difference between this theory and classical theory or conventional transverse shearing theory is the use of trigonometric terms in addition to the usual constant and linear terms that represent the through-the-thickness variation of the displacements. This new theory is applied to cylindrical bending plate problems in Ref. 2 to show that the theory can predict directly the transverse stresses as well as the in-plane stresses. Reference 3 presents results for the buckling of thick, rectangular simply supported aluminum plates in compression showing that transverse shearing flexibility must be included for thicker plates. Reference 3 also presents postbuckling results for long, thin simply supported plates in compression, which show that measureable transverse shear strains occur in the postbuckling range for the plate made of a ±45 deg composite laminate. These results show changes due to transverse shearing in deformations and forces for the plates made of aluminum as well as the ±45 deg composite laminate when the edges are held straight and, as expected, changes for all boundary conditions for transverse shearing force. The purpose of the present paper is to present similar results for the buckling of thick, rectangular simply supported aluminum plates in shear and to determine if the plate postbuckling response for thin plates is sensitive to transverse stiffness for in-plane shear loading. Characteristic curves presenting the average shear stress resultant as a function of the applied displacement are given for a wide variety of in-plane edge conditions.

Analysis

Buckling

Equation (32) of Ref. 1 presents stability equations for transversely isotropic plates where the effects of displacements trigonometric in z, the coordinates through the thickness

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(three-dimensional flexibility theory) as well as linear in z (conventional transverse shearing and classical theories), are taken into account. These equations are solved for simply supported, rectangular aluminum plates loaded in in-plane shear for various length-to-width ratios and thickness ratios using all three theories. Delayed division is the method of solution as described in Ref. 4. The results are obtained using 40 terms in a double Fourier sine series.

Postbuckling

The derivation of the equations to be solved using classical (Kirchhoff) theory has been presented in Refs. 5 and 6. The derivation of equations using conventional transverse shearing theory follows similarly and, accordingly, is not presented in detail here. Using both theories allows one to decide which approximation is needed to provide reasonably accurate results for the range of variables considered.

To help understand the differences between the various theories, it is convenient to start with the more complicated theory and then neglect or change terms to get to the less complicated theory. For conventional transverse shearing theory, the displacements are [from Eq. (24) of Ref. 1]

$$u(x,y,z) = u^o(x,y) + u^a(x,y)(z/h)$$
$$v(x,y,z) = v^o(x,y) + v^a(x,y)(z/h)$$
$$w(x,y,z) = w^o(x,y) \tag{1}$$

and the nonlinear strains corresponding to these displacements are taken to be [from Eq. (25) of Ref. 1]

$$\epsilon_x = u^o_{,x} + \tfrac{1}{2} w^{o\,2}_{,x} + u^a_{,x}(z/h)$$
$$\epsilon_y = v^o_{,y} + \tfrac{1}{2} w^{o\,2}_{,y} + v^a_{,y}(z/h)$$
$$\epsilon_z = 0$$
$$\gamma_{xy} = u^o_{,y} + v^o_{,x} + w^o_{,x}w^o_{,y} + (u^a_{,y} + v^a_{,x})(z/h)$$
$$\gamma_{xz} = w^o_{,x} + (u^a/h)$$
$$\gamma_{yz} = (v^a/h) + w^o_{,y} \tag{2}$$

In classical plate theory, u^a/h and v^a/h are replaced by $-w^o_{,x}$ and $-w^o_{,y}$, respectively, to satisfy the Kirchhoff assumption that lines normal to the undeformed neutral surface remain normal to the neutral surface after deformation.

Guided by Refs. 5 and 6, the form of the unknown displacements are

$$u^o = -U\left(\frac{x}{a} - \frac{1}{2}\right) + u^o_0(y) + u^o_s(y)\sin\frac{2\pi x}{\lambda} + u^o_c(y)\cos\frac{2\pi x}{\lambda}$$

$$u^a = u^a_s(y)\sin\frac{\pi x}{\lambda} + u^a_c(y)\cos\frac{\pi x}{\lambda}$$

$$v^o = v^o_0(y) + v^o_s(y)\sin\frac{2\pi x}{\lambda} + v^o_c(y)\cos\frac{2\pi x}{\lambda}$$

$$v^a = v^a_s(y)\sin\frac{\pi x}{\lambda} + v^a_c(y)\cos\frac{\pi x}{\lambda}$$

$$w^o = w^o_s(y)\sin\frac{\pi x}{\lambda} + w^o_c(y)\cos\frac{\pi x}{\lambda} \tag{3}$$

The transverse displacement w and the bending parts of u and v, which are all exact at buckling, are sinusoidally periodic with half-wavelength λ. The extensional parts of u and v, i.e., u^o and v^o, are sinusoidally periodic with half-wavelength $\lambda/2$ and u^o has an extra linear-in-x term associated with the constant U that is the applied displacement (for compression loading) and, therefore, specified.

The virtual work of the internal forces for a three-dimensional body is

$$\delta\Pi = \int_0^a \int_0^b \int_{-h/2}^{h/2} (\sigma_x\delta\epsilon_x + \sigma_y\delta\epsilon_y + \sigma_z\delta\epsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{xz}\delta\gamma_{xz} + \tau_{yz}\delta\gamma_{yz})\,dx\,dy\,dz \tag{4}$$

For the cases treated here, Hooke's law for the relation of stresses to strains for each lamina is taken to be

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \tag{5}$$

Stress and moment resultants may be defined. Examples are

$$N_y = \int_{-h/2}^{h/2} \sigma_y\,dz = N^o_y + N^s_y\sin\frac{2\pi x}{\lambda} + N^c_y\cos\frac{2\pi x}{\lambda}$$

$$M_y = \int_{-h/2}^{h/2} \sigma_y z\,dz = M^s_y\sin\frac{\pi x}{\lambda} + M^c_y\cos\frac{\pi x}{\lambda}$$

Differential equations and variationally consistent boundary conditions were derived from the virtual work in Ref. 5 or 6 for the conventional transverse shearing theory in much the way as for classical theory. A computer program was then written for this case. The detailed boundary conditions at $y = 0,b$ used were

Classical

$$u^o_s = u^o_c = \begin{Bmatrix} v^o_0 \\ \text{or} \\ N^o_y \end{Bmatrix} = \begin{Bmatrix} v^o_s = v^o_c \\ \text{or} \\ N^s_y = N^c_y \end{Bmatrix}$$

$$= w^o_s = w^o_c = M^s_y = M^c_y = 0$$

Conventional transverse shearing

$$u^o_s = u^o_c = u^a_s = u^a_c = \begin{Bmatrix} v^o_0 \\ \text{or} \\ N^o_y \end{Bmatrix} = \begin{Bmatrix} v^o_s = v^o_c \\ \text{or} \\ N^s_y = N^c_y \end{Bmatrix}$$

$$= w^o_s = w^o_c = M^s_y = M^c_y = 0$$

where the edges were straight if $v^o_s = v^o_c = 0$ in addition to the other conditions and the edges were considered to be unrestrained if $N^s_y = N^c_y = 0$ in addition to the other conditions.

The half-wavelength λ of interest for the infinitely long plates considered here is the one that corresponds to minimum shear load and the solution of interest is on the equilibrium path that gives nonzero deflections.

Results and Discussion

The results obtained in this study for the aluminum plates are based on the thickness $h = 0.1$ in. and with mechanical properties of $E = 10.7 \times 10^6$ psi and $\mu = 0.33$.

Buckling results for an aluminum plate are presented in Fig. 1. The results presented in Fig. 1 show the variation of the buckling stress coefficient with thickness ratio as given by the

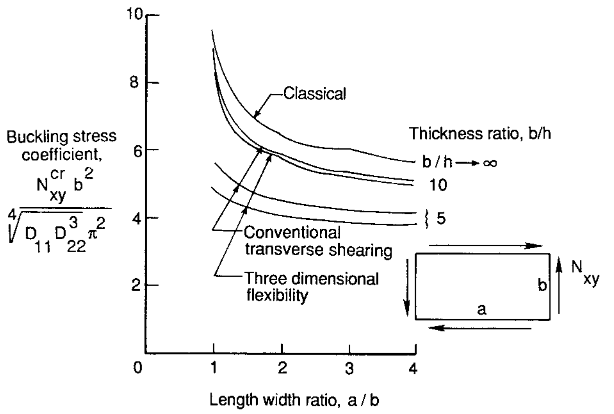

Fig. 1 Effect of thickness on the shear buckling stress coefficient for simply supported aluminum plates.

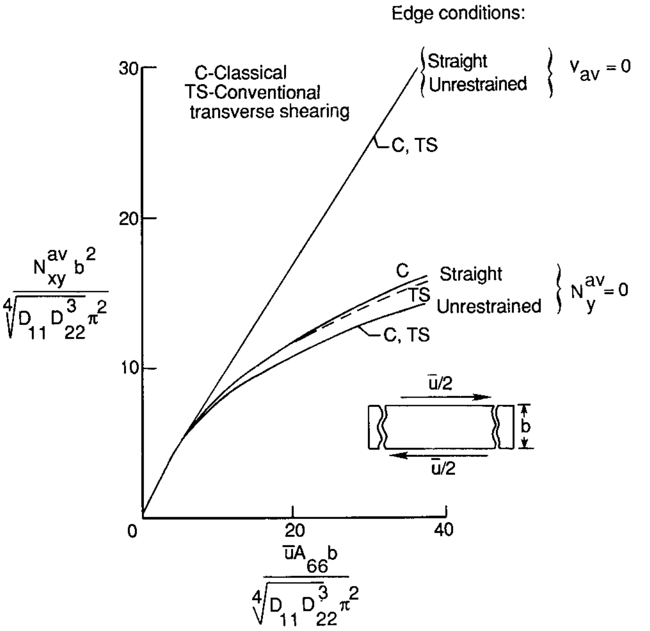

Fig. 2 Characteristic curves for the postbuckling of long, simply supported aluminum plates in shear.

three theories (classical, conventional transverse shearing, and three-dimensional flexibility) for a range of length-to-width ratios. These results show that, for aluminum plates with thicknesses greater than about 5% of the plate width, the effects of transverse shearing should be included in determining the in-plane shear buckling stress. This lower buckling stress would be important in determining the buckling load of heavily loaded shear webs or heavily loaded fuselages with closely spaced stiffeners and the start of postbuckling. However, a closer look at these results shows that, at best, for thickness ratios of interest (for thicker plates), the buckling stresses are near or above the elastic limit and elastic postbuckling studies could serve only as a guide. Accordingly, in order to be of practical use, the postbuckling results presented here will be directed toward plates with thickness ratios b/h greater than 10. Therefore, from the results of Fig. 1, it would seem that only classical theory would be needed; however, for compression (see Ref. 3) it was found that for postbuckling conventional transverse shearing theory was needed to get accurate transverse shear loads and for straight edges to get more accurate displacements and force resultants.

Characteristic curves are presented in Figs. 2 and 3 for the postbuckling in shear of rectangular plates made of aluminum

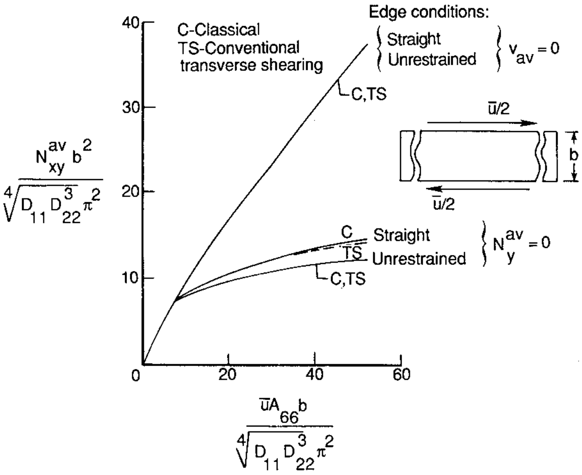

Fig. 3 Characteristic curves for the postbuckling of long, simply supported ±45 deg graphite-epoxy plates in shear.

and ±45 deg graphite epoxy, respectively. The aluminum results are based on the properties given above with the width $b = 10$ in. The ±45 deg graphite-epoxy laminate results are based on the dimensions of $h = 0.1$ in. and $b = 10$ in. and the following properties (the units of D are inch-pounds and the units of A are pounds/inch):

$D_{11} = D_{22} = 0.5186 \times 10^3$ $\qquad A_{33} = 0.59 \times 10^5$

$D_{12} = 0.37291 \times 10^3$ $\qquad A_{12} = 0.44606 \times 10^6$

$D_{66} = 0.40423 \times 10^3$ $\qquad A_{44} = A_{55} = 0.5 \times 10^5$

$A_{11} = A_{22} = 0.62034 \times 10^6$ $\qquad A_{66} = 0.48352 \times 10^6$

The average in-plane shearing stress corresponding to the applied in-plane displacement is plotted in terms of dimensionless parameters for infinitely long plates. Curves were obtained from both theories—the classical and the conventional transverse shearing. The curves give the same results at buckling, but then begin to deviate, with curves representing the four in-plane edge conditions considered with the simple support ($w = M_y = 0$) condition. The in-plane conditions considered are that either the average values of the in-plane displacement or the resultant force are zero ($v_{av} = 0$ or $N_y^{av} = 0$) and that either the edges are straight or unrestrained to move in and out. Identifying the curves in terms of predominately extensional or bending behavior leads to some interesting observations. First, one expects that the plate behavior under increasing extension may involve large strains followed by plasticity and then overall changes in thickness (necking down), but with no unusual effects associated with thickness in the elastic range. A plate in bending undergoes large deflections and folds or kinks, involving large thickness (transverse shearing) effects and plasticity. The present problem involves extensional deformations prior to buckling and bending deformations at buckling. From the high slopes of the postbuckling curves, it seems safe to say that the curves presenting results for $v_{av} = 0$ involve predominately extensional behavior (such as diagonal tension). The curve with the lowest slope represents the weakest in-plane condition and therefore must be associated more closely with bending. The buckle wavelengths of the plates with higher postbuckling stiffness are lower than the wavelength at the critical load and the wavelengths of the plates of lower postbuckling stiffness are higher. The results for $N_y^{av} = 0$ involve predominately bending behavior and, for the stiffer in-plane conditions (straight edges), the wavelengths are not as long as for the weaker in-plane conditions (unrestrained), but still larger than the wavelength at buckling.

Also, small effects due to transverse shearing begin to appear, indicating that transverse shearing will become important as the applied shearing displacement is increased. The present analysis has the most transverse shearing effects when the edges are straight, but free of in-plane stress N_y^{av} on the average. This was true also for compression loading (see Ref. 3). Here, the characteristic curves were affected; there the effect did not show up on the characteristic curves.

Conclusions

Results presented here for aluminum plates show that theories which include transverse shearing flexibility are required to obtain buckling loads for thicker plates in shear. When the shear loads go beyond buckling, there is a considerable range of results possible for simply supported thin plates with different in-plane boundary conditions representing a range of behavior from mostly extensional to mostly bending. Transverse shear effects are more important for thin plates with straight edges and with mostly bending behavior as the shear loading is increased. Thin plates made of a ±45 deg graphite-epoxy laminate have less stiffness in the postbuckling range than aluminum plates of the same overall dimensions for both compression and shear loading. Beyond buckling, the analysis of thin plates in compression has also been shown to require theories that include transverse shearing flexibility for the boundary condition free of in-plane stress, but with the edges held straight. The present results show that classical theory is valid for shear for applied displacements up to about seven times the critical value or more, depending on the in-plane conditions.

References

[1]Stein, M., "Nonlinear Theory for Plates and Shells Including the Effects of Transverse Shearing," *AIAA Journal,* Vol. 24, Sept. 1986, pp. 1537–1544.

[2]Stein, M. and Jegley, D. C., "Effects of Transverse Shearing on Cylindrical Bending, Vibration and Buckling of Laminated Plates," *AIAA Journal,* Vol. 25, Jan. 1987, pp. 123–129.

[3]Stein, M. and Bains, N. J. C., "Postbuckling Behavior of Longitudinally Compressed Orthotropic Plates with Three-Dimensional Flexibility," AIAA Paper 86-0976, May 1986.

[4]Thurston, G. A. and Bains, N. J. C., "Solution of the Symmetric Eigenproblem $AX = \lambda BX$ by Delayed Division," NASA TP 2514, March 1986.

[5]Stein, M., "Postbuckling of Long Orthotropic Plates in Combined Shear and Compression," *AIAA Journal,* Vol. 23, May 1985, pp. 788–794.

[6]Stein, M., "Analytical Results for Post-Buckling Behavior of Plates in Compression and in Shear," *Aspects of the Analysis of Plate Structures,* edited by D. J. Dawe et al., Clarendon, Oxford, England, 1985, pp. 205–223.